On the Stability of a Market-Oriented Cloud Computing Model with Time-Varying Workloads

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1. Introduction
2. About Passivity Analysis
3. Effect of Equilibrium Points in Stability
4. Effect of Time-Varying $w(k)$
5. Simulation Results
6. Conclusions



1 Introduction

2 About Passivity Analysis

3 Effect of Equilibrium Points in Stability

4 Effect of Time-Varying $w(k)$

5 Simulation Results

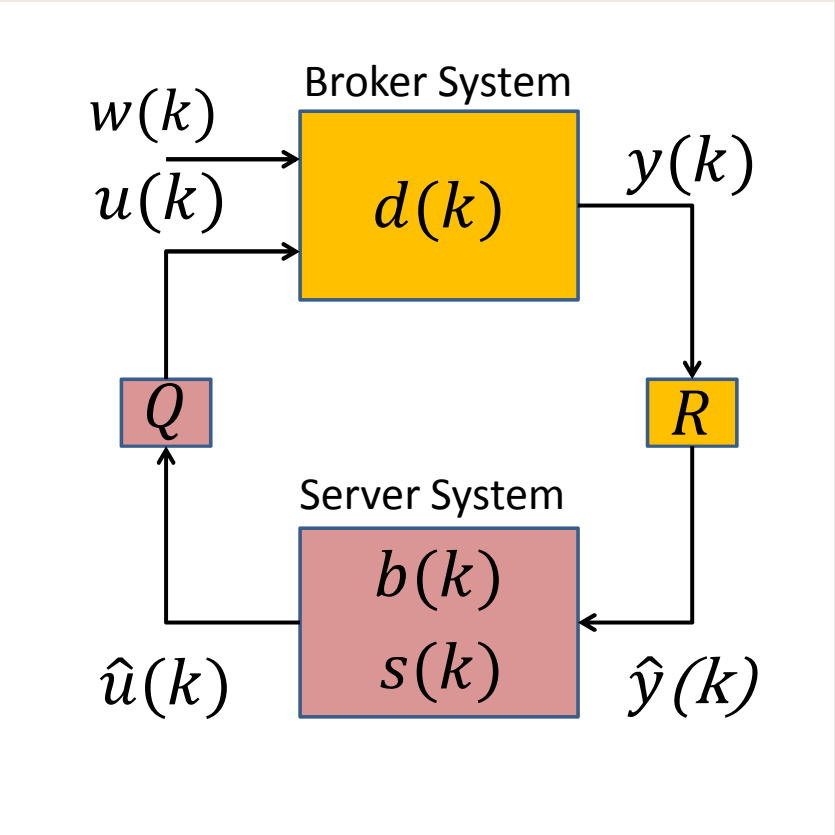
6 Conclusions



1 Introduction

- Original Work [Lemmon, 2012]
- Market-Oriented Cloud Model [Lemmon, 2012]
- Contributions

- A passivity framework to manage cloud computing systems from a market-oriented point of view is proposed.



Original work [Lemmon, 2012]

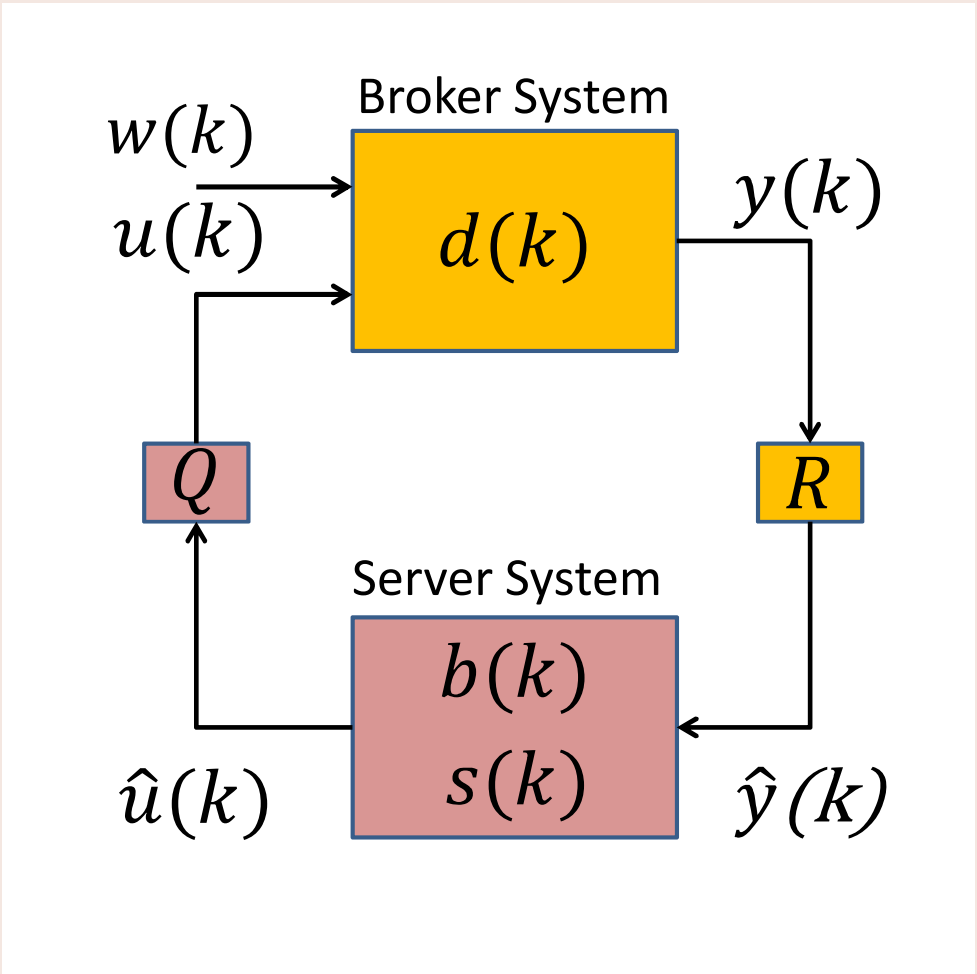
- The routing of the consumer's workload into the cloud is described by a market-oriented discrete-time model.
- Servers communicate with brokers letting them know how busy they are.
- Brokers distribute the consumer's workload to be processed by the servers based on the current status of the servers.
- This framework takes advantage of the passivity inherited by a system formed by interconnected passive subsystems.



1 Introduction

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Market Oriented Cloud [Lemmon, 2012]



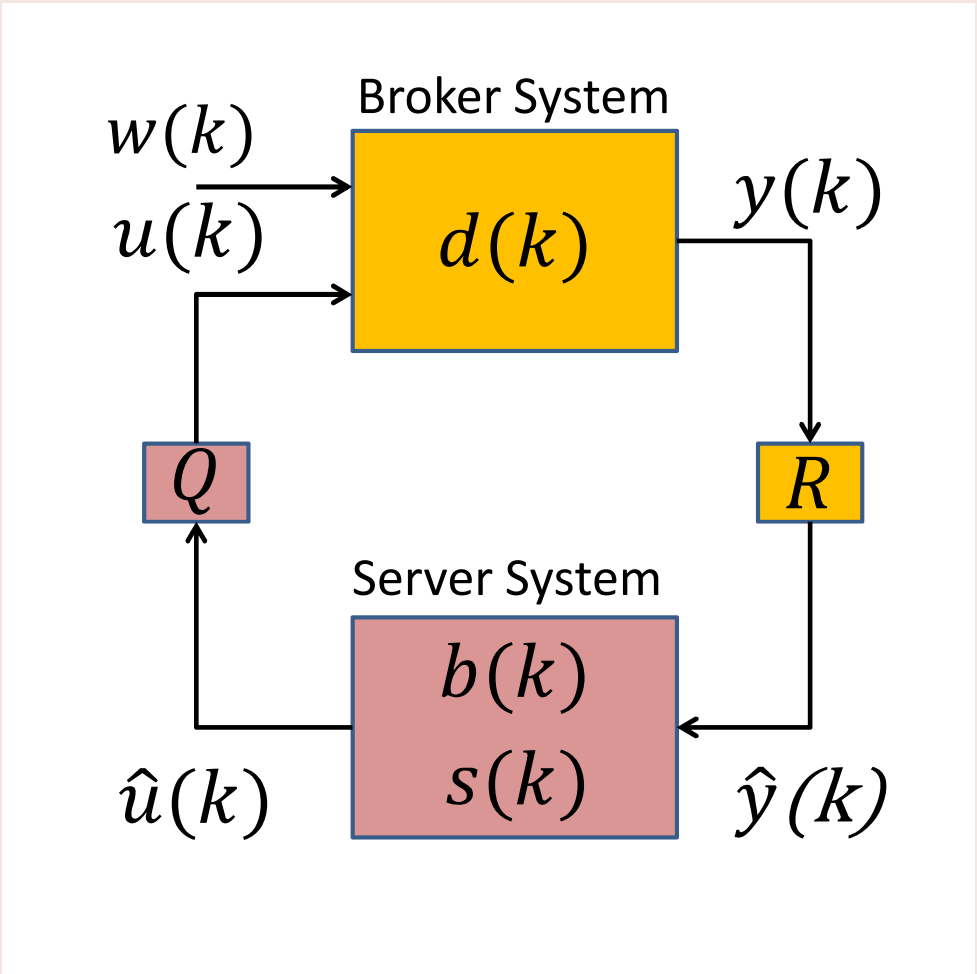
Model of the brokers [Lemmon, 2012]

$$d_j(k+1) = \left[(1-\beta_{1j})d_j(k) + \beta_{1j}w_j(k) - \beta_{2j}u_j(k)\right]^+,$$

$$y_j(k) = \min\{w_j(k), d_j(k)\}.$$

$d_j(k)$ with $j = 1, 2, \ldots, N$, *maximum dispatch level* at time instant $k \in \mathbb{Z}^+$. $\beta_{1j} \in (0,1) \subset \mathbb{R}$ and $\beta_{2j}(0,\infty) \subset \mathbb{R}$ and $[\cdot]^+ = \max(\cdot, 0)$. $y_j(k)$, output of the system.

Market Oriented Cloud [Lemmon, 2012]



Model of the servers [Lemmon, 2012]

$$\begin{aligned} b_i(k+1) &= [b_i(k) + \hat{y}_i(k) - s_i(k)]^+, \\ s_i(k+1) &= \min\{\bar{s}_i, (1-\sigma_i)s_i(k) + b_i(k) + \hat{y}_i(k)\}. \end{aligned}$$

$b_i(k)$, amount of pending workload. $s_i(k)$, maximum amount of workload that the i-th server can process. $\bar{s}_i$, upper bound of $s_i(k)$.

$$\hat{u}_i(k) = 2\sigma_i b_i(k) + 2\sigma_i s_i(k).$$

$\sigma_i \in (0, 1)$, a control gain.



The Model [Lemmon, 2012]

- The broker system is output strictly passive.
- The server system is passive.
- From passivity theory the origin of the interconnected system with $w(k) = 0$ is asymptotically stable.

1 Introduction

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Contributions

- We provide a mathematical justification to the use of Passivity theory.
- We proof that an additional condition is required to guarantee the asymptotic stability of the system and proceed to formulate it.
- We proof that the system is Input-to-State Stable (ISS) in presence of time-varying consumer's workload.

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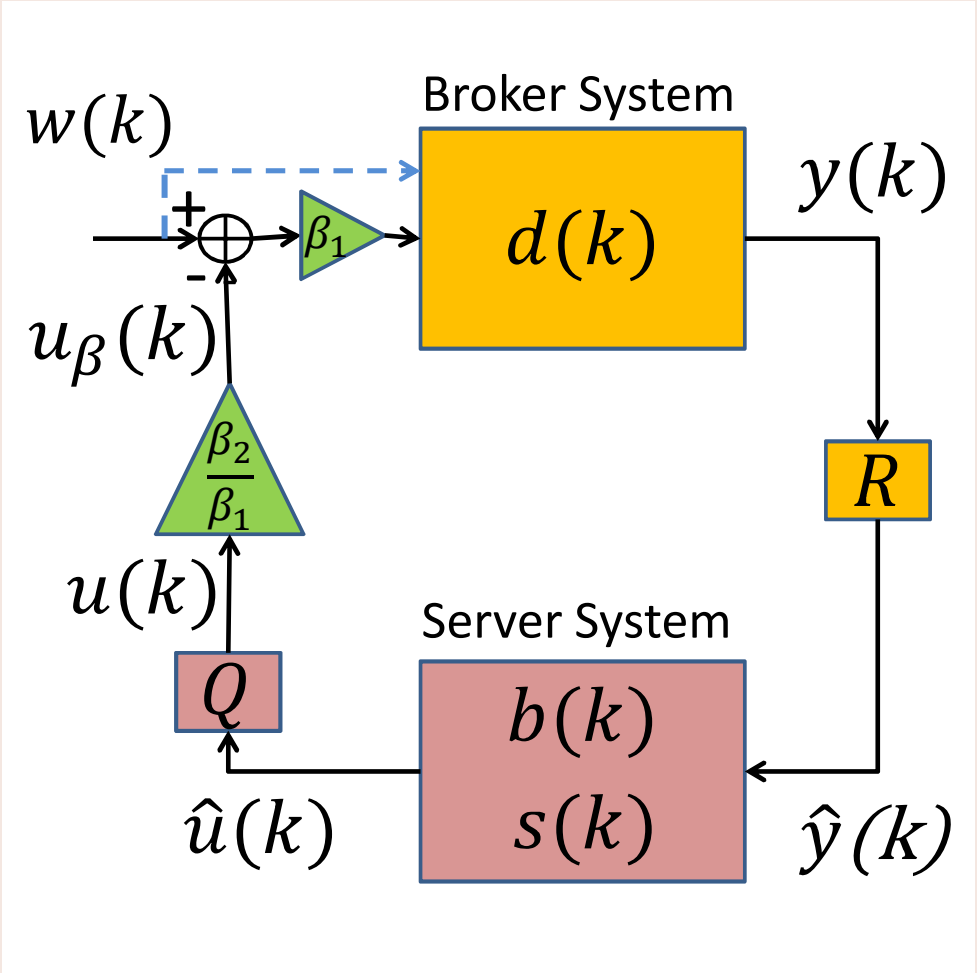


2 About Passivity Analysis

- 1st Theoretical Result



Market Oriented Cloud



Proposition

Consider the state-space dynamics of the j-th broker defined by (1) and (2). For any initial condition $d(k_0)$ such that $d(k_0) > w(k) = w > 0$ with $w \in \mathbb{R}^+$ constant, there exists $N < \infty$, $N \in \mathbb{Z}^+$ such that
$y(k) = \min\{w, d(k)\} = d(k), \forall k \geq k_0 + N.$

Remember that...

$$d_j(k+1) = \left[(1-\beta_{1j})d_j(k) + \beta_{1j}w_j(k) - \beta_{2j}u_j(k)\right]^+, \quad (1)$$

$$y_j(k) = \min\{w_j(k), d_j(k)\}. \quad (2)$$

Remark

Notice that for the case $w(k) = w = 0$ the foregoing Proposition does not apply, since from [Lemmon, 2012], $d(k) \to 0$ as $k \to \infty$ asymptotically, i.e., in infinite time.

Remark

In the foregoing Proposition we do not consider the case where the projection of $d(k)$ is active because the main assumption is $d(k) > 0$.



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3 Effect of Equilibrium Points in Stability

- Counter-example
- 2nd Theoretical Result

Counter-example: A simple case

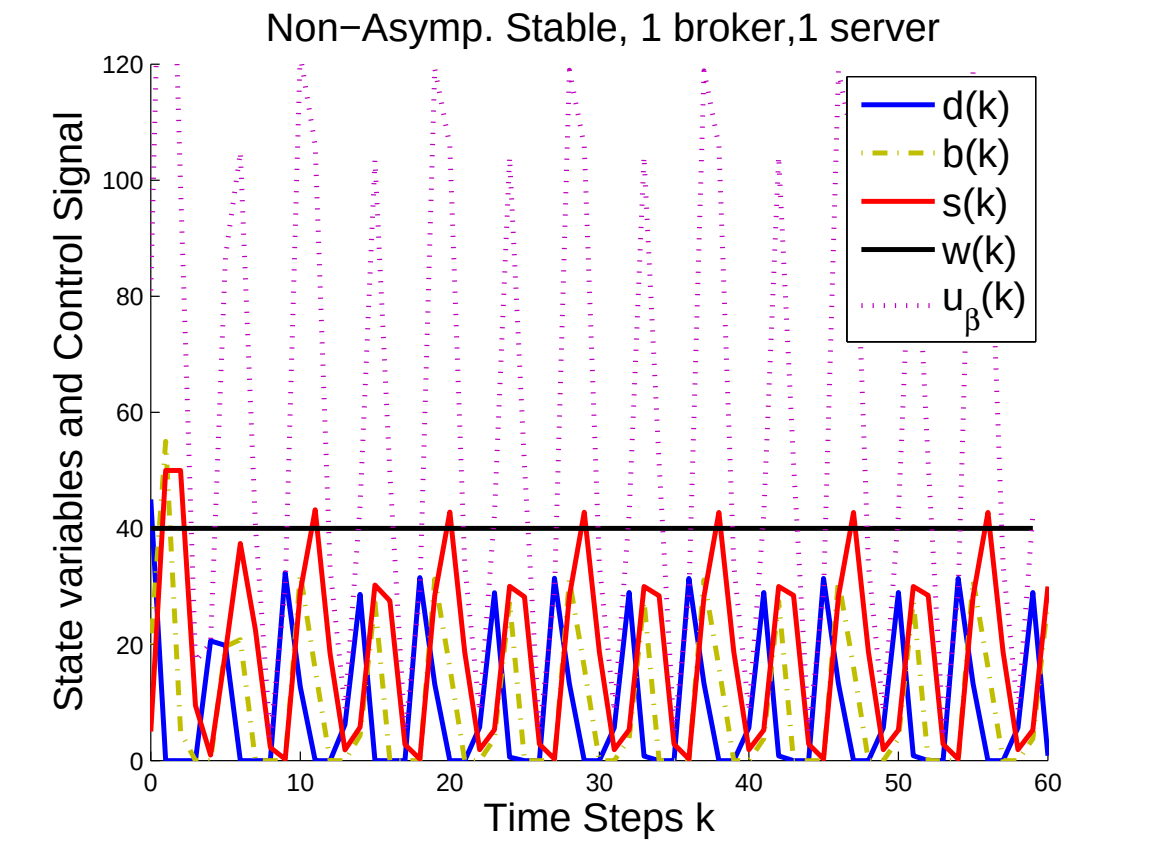
Let us simulate one broker and one server with $\beta_1 = 0.9$, $\beta_2 = 0.9$, $w(k) = w = 40$, $\bar{s} = 50$, initial conditions $d(k_0) = 45, b(k_0) = 20, s(k_0) = 5$ and

$$\hat{u}_1(k) = 2\sigma b(k) + 2\sigma s(k) + \sigma \hat{y}(k).^a \tag{3}$$

with $\sigma = 0.9$, we get the following plot...

[a] See Appendix A in the paper for a justification for using (3)

Counter-example: A simple case



3 Effect of Equilibrium Points in Stability

- Counter-example
- 2nd Theoretical Result

Proposition

An additional condition to guarantee the asymptotic stability of the feedback connection between the broker system given by (1) and (2), and the server system given by (5), (6) and (3) is,

$$0 < \beta_2 u(k) \leq \beta_1 w(k). \tag{4}$$

Remember that...

$$b_i(k+1) = [b_i(k) + \hat{y}_i(k) - s_i(k)]^+, \tag{5}$$

$$s_i(k+1) = \min\{\bar{s}_i, (1-\sigma_i)s_i(k) + b_i(k) + \hat{y}_i(k)\}. \tag{6}$$



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4 Effect of Time-Varying $w(k)$

- 3rd Theoretical Result

Proposition

Given the feedback connection in Fig. 2 defined by the broker system with dynamics (1) and (2) with $0 < \beta_1 < 1, \ \beta_1 \in \mathbb{R}$ *and* $0 < \beta_2, \ \beta_2 \in \mathbb{R}$, *and the server system with dynamics (5),(6) and (3) with* $\sigma \in (0, 1) \subset \mathbb{R}$, *the resulting system is ISS.*

Furthermore, if $0 < \beta_2 u(k) < \beta_1 w(k)$, *the system tracks the equilibrium points*

$$d_{eq} = w(k) - \frac{\beta_2}{\beta_1} u(k), \ b_{eq} = 0, \ s_{eq} = y(k).$$

asymptotically.

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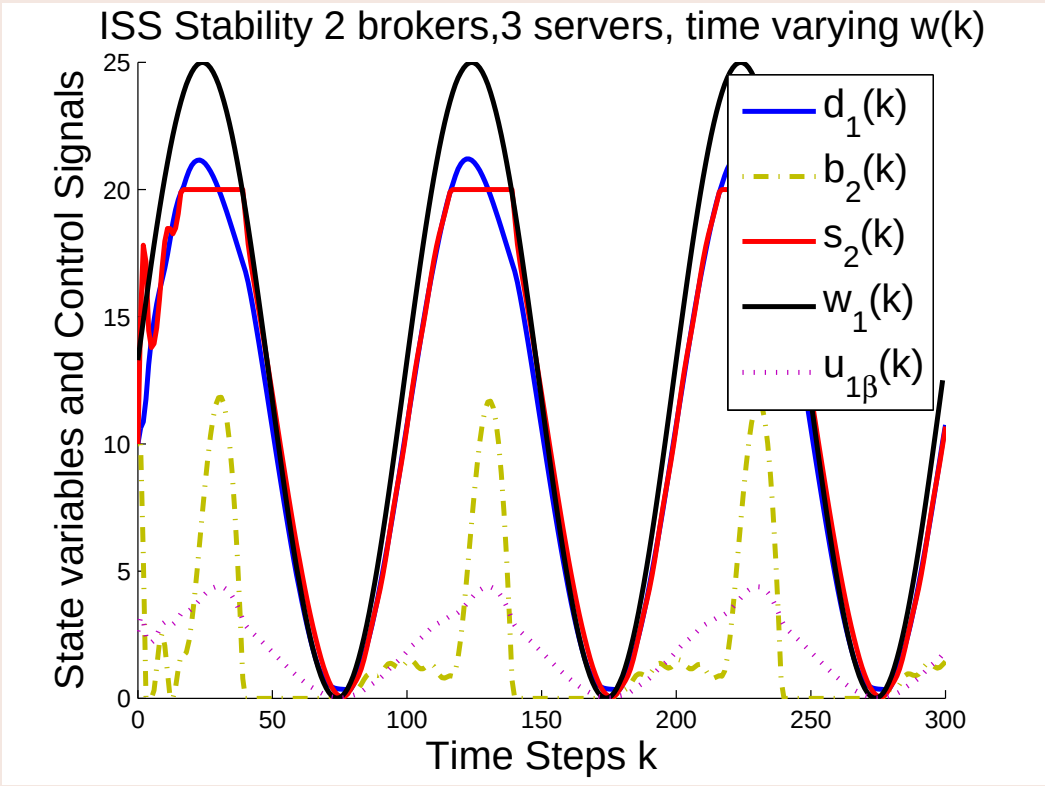
6 Conclusions



Simulation Parameters

- 2 brokers and 3 servers,
- $\beta_1 = 0.95$, $\beta_2 = 0.1$, $\sigma = 0.5$,
- $\bar{s} = 20$,
- $w(k) = 12.5 + 12.5 \sin(\frac{2\pi k}{100})$

Simulation Results

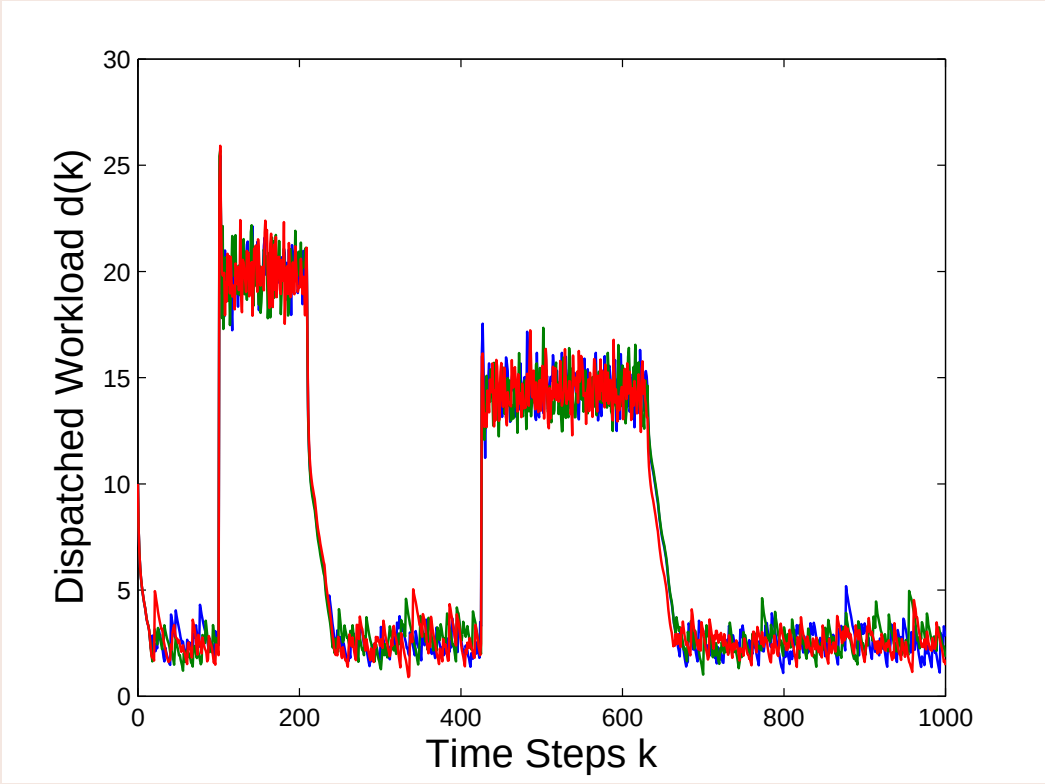


Simulation Parameters

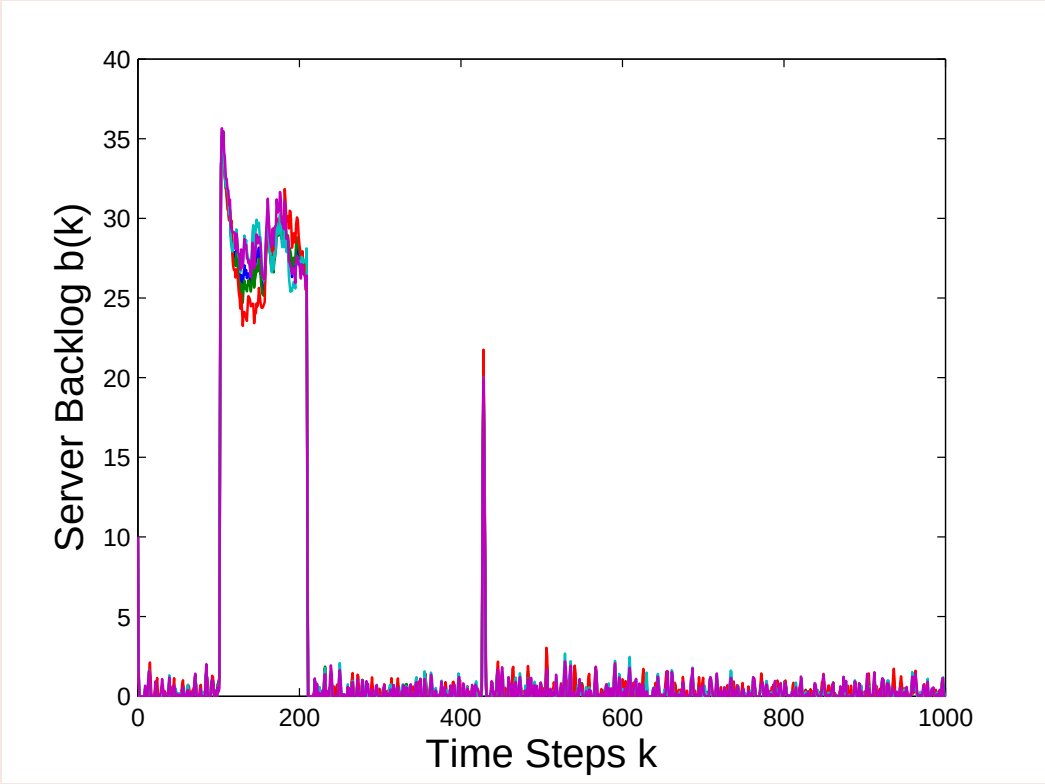
- 3 brokers and 5 servers,
- $\beta_1 = 0.95$, $\beta_2 = 0.1$, $\sigma = 0.5$,
- $\bar{s} = 20$,
- Consumer's workload modeled as a Gaussian white noise with mean $\mu = 2$ and variance $\sigma_g^2 = 1$,
- At time step $k = 100$ the mean of $w(k)$ is changed to $\mu = 25$ and at $k = 210$ it goes back to $\mu = 2$.
- At $k = 425$ the mean goes to $\mu = 16$ and at $k = 632$ it returns to $\mu = 2$.



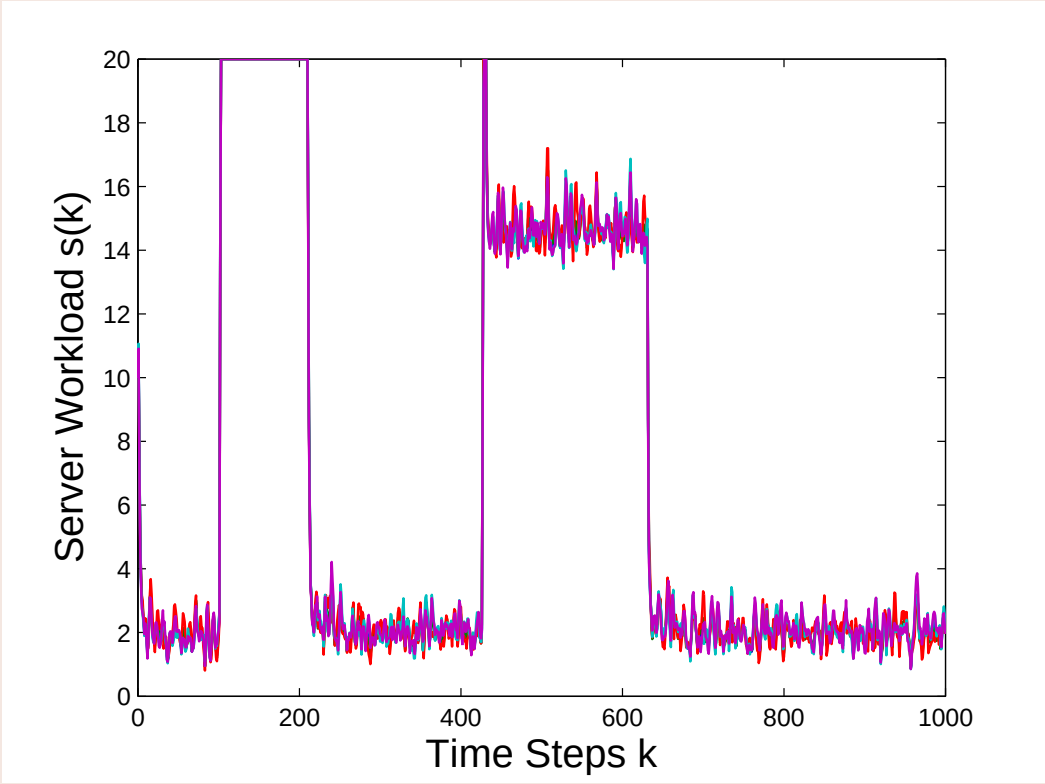
Simulation Results



Simulation Results



Simulation Results



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- We have presented an in depth analysis of the passivity framework introduced in [Lemmon, 2012] for power control and response time management in the cloud.
- We have formally provided an additional condition for the asymptotic stability of the market-oriented cloud model.
- Additional assumptions to justify the use of the passivity approach to guarantee asymptotic stability have been provided.
- We have formally proven that the the cloud model is ISS in the presence of time-varying consumer's workload vectors.



Questions ?

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References



Lemmon, M. D. (2012).
Towards a passivity framework for power control and response time management in cloud computing.
In *Proc. of 7th Intl. Workshop on Feedback Computing*, San Jose, CA.

Proof Proposition 1

Let us define a new state variable

$$d_0(k) = d(k) - w(k) + \frac{\beta_2}{\beta_1} u(k), \tag{7}$$

therefore, we obtain the new dynamical equation,

$$d_0(k+1) = (1-\beta_1) d_0(k), \ \ \beta_1 \in (0,1). \tag{8}$$

Now, let us propose the following Lyapunov function candidate,

$$V_3(d_0) = d_0^2(k),$$

Proof Proposition 1

The first difference gives,

$$\Delta V_3 = (-1 + (1 - \beta_1)^2) d_0^2(k) \leq 0, \tag{9}$$

and the origin of (8) is asymptotically stable.
Let us assume an initial condition $d_0(k) > w$. Furthermore, from (9) we know that for any $d_0(k) > 0$ there exists $\eta \in \mathbb{R}^+$ such that

$$\Delta V_3 < (-1 + (1 - \beta_1)^2) d_0^2(k) < -\eta,$$

therefore,

$$V_3(k+1) - V_3(k) < -\eta,$$

Proof Proposition 1

Solving the recurrence equation we get,

$$V_3(k) \leq V_3(k_0) - (k - k_0)\eta, \tag{10}$$

If we take any feasible $\delta \in \mathbb{R}^+$ in the trajectory of $d(k)$ such that $d(k_0) > \delta > 0$ we get $V_3(\delta) = \delta^2$. Therefore, if starting from the initial condition $d(k_0)$ we arrive at $d(k) = \delta$ for some k, we get from (10) that,

$$\delta^2 \leq V_3(d(k_0)) - (k - k_0)\eta,$$

Proof Proposition 1

Therefore,

$$k \leq k_0 + \frac{V_3(k_0) - \delta^2}{\eta} < \infty.$$

Then, the number of steps required to go from any initial state $d_0(k_0) > 0$ to another state $d_0(k) > 0$ in the trajectory of the solution of (8) is finite. From (7) we conclude that starting from an initial state $d(k_0)$, there exists $N \in \mathbb{Z}^+, N < \infty$ such that $0 < d(k) \leq w,\ \forall k > k_0 + N$, therefore,
$y(k) = \min\{w, d(k)\} = d(k) > 0,\ \forall k > k_0 + N.$ □

Proof Proposition 2

Equilibrium points of the j-th broker and the j-th server,

$$\begin{aligned} d_{eq} &= w(k) - \frac{\beta_2}{\beta_1} u(k), \\ b_{eq} &= 0, \\ s_{eq} &= y(k). \end{aligned}$$

Now, let us assume

$$0 < w(k) < \frac{\beta_2}{\beta_1} u(k), \tag{11}$$

but from the projection in (1) the equilibrium point $d_{eq} \geq 0$, which contradicts (11). □

Proof Proposition 3

Let us study the broker system defined by (1) and (2), and let us define,

$$e_1 = w(k) - \frac{\beta_2}{\beta_1} u(k),$$

then, the system (1) may be rewritten as,

$$d(k+1) = \left[(1-\beta_1)d(k) + \beta_1 e_1\right]^+ . \tag{12}$$

Assuming no active projection in (12), we note that the system is Linear Time Invariant (LTI). Based on [Lemmon, 2012] the origin with $e_1(k) = 0$ is asymptotically stable. Since the system is LTI we conclude that it is Bounded-Input-Bounded-Output (BIBO) stable as well.

Proof Proposition 3

Similarly, the origin of the server system given by (5), (6) and (3) was proven to be stable with $\hat{y}(k) = 0$. Assuming no active projection, the system is LTI, therefore, it is BIBO stable. Assuming an active projection in the server system the system matrix **A** becomes,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ \sigma_i & 1 - \sigma_i \end{pmatrix},$$

and the system is LTI, therefore it is BIBO stable.

Proof Proposition 3

Furthermore, with $y(k) < \infty$ and $\hat{\mathbf{y}}(k) = \mathbf{R}(k)\mathbf{y}(k)$ where all the entries of $\mathbf{R}(k)$, namely, $R_{ij}(k) \in [0, 1]$, then $\hat{y}(k) < \infty$.
Since $\mathbf{u}(k) = \mathbf{Q}(k)\hat{\mathbf{u}}_1(k)$ where the entries of $\mathbf{Q}(k)$, namely, $Q_{ji}(k) \in [0, 1]$, then $u(k) < \infty$.
Finally, since we know that $\frac{\beta_2}{\beta_1}u(k) < \infty$ and $w(k) < \infty$, then $w(k) - \frac{\beta_2}{\beta_2}u(k) < \infty$, and the system is ISS.
If in addition, (4) is satisfied, then the system tracks the equilibrium points asymptotically. □

Appendix A: Recalculation of $\hat{u}(k)$

Assuming that projection in (5) is inactive the first difference of the corresponding storage function is,

$$\begin{aligned}
\Delta V_2 &\leq \xi^T(k+1)P\xi(k+1) - \xi^T(k)P\xi(k) \\
&= \xi^T(k)(A^TPA - P)\xi(k) + 2\xi^TA^TPB\hat{y}(k) \\
&\quad + \hat{y}(k)B^TPB\hat{y}(k) \\
&= 2\xi^TA^TPB\hat{y}(k) + \hat{y}(k)B^TPB\hat{y}(k) \\
&= 2\sigma b(k)\hat{y}(k) - \sigma s(k)\hat{y}(k) + \sigma\hat{y}^2(k) \\
&\leq 2\sigma b(k)\hat{y}(k) + 2\sigma s(k)\hat{y}(k) + \sigma\hat{y}^2(k) \\
&= \hat{u}_1(k)\hat{y}(k).
\end{aligned}$$

$$\Rightarrow \hat{u}_1(k) = 2\sigma b(k) + 2\sigma s(k) + \sigma\hat{y}(k).$$

Appendix A: Recalculation of $\hat{u}(k)$

Now, using the same storage function but assuming that the projection in (5) is active we obtain,

$$\begin{aligned}
\Delta V_2 &\leq \sigma(\sigma-1)(s(k)-b(k))^2 + 2\sigma b(k)\hat{y}(k) \\
&\quad +2\sigma s(k)\hat{y}(k) \\
&\leq 2\sigma b(k)\hat{y}(k) + 2\sigma s(k)\hat{y}(k) + \sigma\hat{y}^2(k) \\
&= \hat{u}_1(k)\hat{y}(k).
\end{aligned}$$